FRACTAL SPACES AND BI-LIPSCHITZ EMBEDDINGS INTO BANACH SPACES

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Abstract

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In this thesis, we examine the geometry of fractals and metric spaces. We study the question of which fractal metric spaces can be embedded into Banach spaces up to a certain distortion.

Our main focus is on a metric space introduced by Urs Lang and Conrad Plaut in "Bi-Lipschitz Embeddings of Metric Spaces into Space Forms," which we refer to as the Diamond Graph Fractal. By modifying the construction methods defined by Lang and Plaut, we develop a Generalized Diamond Graph Fractal and study whether the space converges in the Gromov-Hausdorff distance, satisfies the doubling property, and whether it can be Bi-Lipschitzly embedded into certain Banach spaces with given properties. Our approach to the Bi-Lipschitz embedding problem is to generalize the argument of Lang and Plaut, which involves the quadrilateral inequality, a property of namely Hilbert space.

In addition, we also study and explain an argument in the paper "On the Geometry of the Countably Branching Diamond Graphs" by Florent Baudier et. Al., which involves a related class of graphs and "asymptotic midpoint uniform convexity", a property that the norm of certain Banach spaces, including Hilbert spaces, can satisfy. Our goal, by comparing these two arguments, is to better understand the properties of Banach spaces and how these properties interact with the geometry of certain fractal metric spaces.

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Contents

1 Introduction

Metric embeddings have become a subject of interest in computer science, analysis, geometry, and topology. Here, we continue the study of metric embeddings by focusing on geometric objects and their ability, or lack thereof, to embed reasonably into other spaces, namely Banach spaces. The main problem isn't whether a geometric object can be put into a target space, rather if the geometric object can maintain its sense of distance with respect to the metric of the target space. Therefore we use Lipschitz, or by extension Bi-Lipschitz, embeddings to determine whether distances between the image of two points in the target space is bounded above, and below in the case of Bi-Lipschitz embeddings, by some constant multiple of the distance of the points in the preimage.

The objects that are of particular interest to us are sequences of graphs, where the sequence is self-similar in that each graph contains copies of the previous graphs in the sequence, hence creating fractal spaces. Using Bi-Lipschitz embeddings, we wish to determine what properties either the preimage or the target space gain from what is known about the other. In particular, we consider properties the norm of a Banach space might have that could prevent a graph from being Bi-Lipschitzly embedded into that space. We define norm properties of interest to us in section 2, as well as provide examples to explain how such properties relate to certain Banach spaces.

In some cases, we are able to build fractal metric spaces by taking Gromov-Hausdorff limits of these graphs. We study the non-embeddability of these limits and sequences in sections 3 and 4 following a generalization of Lang and Plaut's argument in [7] and an edited exhibition of theorems provided by Baudier et Al. in [1] respectively.

1.1 Background

Our first example is a generalization of what is described as the "Diamond Graph" provided by Urs Lang and Conrad Plaut [7]. Here Lang and Plaut establish a Gromov-Hausdorff convergent sequence of compact, geodesic spaces $X_0, X_1, ...$, where X_0 is defined to be an edge isometric to the interval [0,1], and for $i = 1, 2, 3, ...$ they take 6 copies of X_{i-1} and rescale them by 1/4 [7]. Then attach four copies cyclically by identifying pairs of endpoints, and then attaching the remaining two copies to two opposite points in order to form X_i [7]. After creating this graph, Lang and Plaut prove that the Diamond graph is doubling, which would suggest that it could be Bi-Lipschitzly embedded into spaces which are also doubling, such as Euclidean space and Hilbert space [7]. However, Lang and Plaut prove that the Diamond Graph cannot be Bi-Lipschitzly embedded into Banach spaces whose norm satisfies the quadrilateral inequality, such as Euclidean space and Hilbert space, even though the Diamond Graph is a doubling space, hence showing that having the doubling property is a necessary, but not sufficient, condition to prove that an object has a Bi-Lipschitz embedding into other doubling Banach spaces [7].

And so by changing how each X_i for $i = 1, 2, 3, ...$ is created to a more general form, we determine through similar procedure that our "Generalized Diamond Graph" is doubling, but cannot Bi-Lipschitzly embed into Banach spaces whose norm has the quadrilateral inequality. We also determine the rate at which the bounds of such a Bi-Lipschitz embedding expand per successive copies of the Generalized Diamond Graph. To note, Theorem 10.2 provided by Cheeger and Kleiner in [2] also provides proof that the Generalized Diamond Graph, along with a wider variety of graphs, does not Bi-Lipschitzly embed in a space which satisfies the quadrilateral inequality. We provide here a proof that is easier to approach, though more specific in its application.

After introducing the Generalized Diamond Graph, we present an exhibition of Theorem 4.1 of the paper *On the Geometry of the Countably Branching Diamond Graph* provided by Baudier et Al. [1]. In section 4 of this paper, Baudier et Al. begin by defining the Barycenter and Midpoint of a set and then use the asymptotically midpoint uniform convexity property of a norm provided by [4] to show that the sequence of graphs, (G_k^ω), does not Bi-Lipschitly embed into spaces whose norm satifies asymptotic midpoint uniform convexity [1]. We note here, and with more detail in Section 4, that we modify the argument provided by Baudier et Al. [1].

1.2 Definitions

1.2.1 Embeddings

Definition 1.1 (Lipschitz Mapping and Bi-Lipschitz Embedding)**.** Let (X, d_X) and $(X', d_{X'})$ be metric spaces and define a mapping $f : X \to X'$. Let $\lambda \geq 0$. We say f is a Lipschitz Mapping if

$$d_{X'}(f(x), f(y)) \leq \lambda d_X(x, y)$$

holds for all $x, y \in X$. Furthermore, we say f is a Bi-Lipschitz Embedding if, for some $\lambda \geq 1$,

$$\lambda^{-1} d_X(x, y) \leq d_{X'}(f(x), f(y)) \leq \lambda d_X(x, y)$$

holds for all $x, y \in X$.

Definition 1.2 (Y-Distortion of X, $c_Y(X)$)**.** Presented by Baudier et Al. in [1]. Let (X, d_X) and (Y, d_Y) be two metric spaces. Then

$$c_Y(X) := \inf\{D \geq 1 | s \cdot d_X(x, y) \leq d_Y(f(x), f(y)) \leq D \cdot s \cdot d_X(x, y)$$
$$\text{for some } f : X \to Y, s > 0, \text{ for all } x, y \in X\}$$

and we denote $c_Y(X)$ as the Y-distortion of X. If there is no such map, f, for any $s > 0, D \geq 1$, then we say $c_Y(x) = \infty$. Informally, the Y-Distortion of X is the best possible constant factor for which Y can be embedded Bi-Lipschitzly in X.

1.2.2 Banach Space Properties

Definition 1.3 (Banach Space)**.** A Banach space, B, is a complete normed vector space with a norm denoted $\| \cdot \|_B$. The unit sphere of B is denoted S_B.

Definition 1.4 (l_p space)**.** Let $1 \leq p < \infty$. We define the space l_p as

$$l_p = \{(x_1, x_2, x_3, ...) : x_i \in \mathbb{R} \text{ for } i = 1, 2, 3, ... \text{ and } (\sum_{i=1}^{\infty} |x_i|^p)^{1/p} < \infty\}$$

with norm $\| \cdot \|_{l_p} := (\sum_{i=1}^{\infty} |x_i|^p)^{1/p}$ and for $x, y \in l_p$, the distance function $d_{l_p}(x, y) := (\sum_{i=1}^{\infty} |x_i - y_i|^p)^{1/p}$. A limiting case of l_p space is

$$l_\infty := \{(x_1, x_2, x_3, ...) : x_i \in \mathbb{R} \text{ for } i = 1, 2, 3, ... \text{ and } \sup_{i \in \mathbb{N}} |x_i| < \infty\}$$

with $\| \cdot \|_{l_\infty} := \sup_{i \in \mathbb{N}}\{|x_i|\}$ and distance function for $x, y \in l_\infty$ $d_{l_\infty}(x, y) := \sup_{i \in \mathbb{N}}\{|x_i - y_i|\}$.

Definition 1.5 (Quadrilateral Inequality)**.** Presented by Lang and Plaut in [7]. Let X be a metric space with metric d_X. We say X satifies the Quadrilateral Inequality if the following inequality holds:

$$\sum_{i=0}^{4}(d_X(x_i, x_{i+1}))^2 \geq (d_X(x_0, x_2))^2 + (d_X(x_1, x_3))^2$$

for all $x_0, x_1, x_2, x_3, x_4 = x_0 \in X$.

Definition 1.6 (λ-Barycenter of X and the δ-approximate Midpoint Set)**.** Presented by Baudier et Al. in [1]. Let X be a Banach space, $x, y \in X$ $\delta \in (0,1)$, and $\lambda \in (0,1)$. The λ-Barycenter of X, denoted $\text{Bar}_\lambda(x, y, \delta)$, is defined as

$$\text{Bar}_\lambda(x, y, \delta) = \left\{ z \in X : \max\left\{ \frac{d_X(x,z)}{\lambda}, \frac{d_X(z,y)}{1-\lambda} \right\} \leq (1+\delta)d_X(x,y) \right\}.$$

As a particular instance of $\text{Bar}_\lambda(x, y, \delta)$, the δ-approximate Midpoint Set (or just Midpoint set), denoted as $\text{Mid}(x, y, \delta)$, is defined as

$$\text{Mid}(x, y, \delta) = \text{Bar}_\lambda(x, y, \delta) \text{ when } \lambda = \frac{1}{2}.$$

Definition 1.7 (Uniform Convexity)**.** Presented by Clarkson in [3]. Let X be a normed vector space. We say X is Uniformly Convex if for every $\epsilon > 0$ there exists a $\delta > 0$ such that for any two vectors $x, y \in X$ with $\|x\| = \|y\| = 1$, $\|x - y\| \geq \epsilon$ implies that $\|\frac{x+y}{2}\| \leq 1 - \delta$. Equivalently, as proven by Laakso in [6], X is Uniformly Convex if and only if, $\sup_{x \in S_X} \text{diam}(\text{Mid}(-x, x, \delta)) \to 0$ as $\delta \to 0$.

Definition 1.8 (Kuratowski measure of noncompactness, α)**.** Presented by Dilworth et Al. in [4]. Let S be a bounded subset of a metric space and $\epsilon > 0$. We define the Kuratowski measure of noncompactness of S, denoted $\alpha(S)$, as the infimum of all $\epsilon > 0$ such that S can be covered by a finite number of sets of diameter less than ϵ.

Definition 1.9 (Asymptotic Midpoint Uniform Convexity (AMUC))**.** Presented by Dilworth et Al. in [4]. Let Y be a Banach space, $y \in S_Y$, and $\delta > 0$. We say the norm of Y is Asymptotically Midpoint Uniformly Convex, or AMUC, if

$$\lim_{\delta \to 0} \sup_{y \in S_Y} \alpha(\text{Mid}(-y, y, \delta)) = 0.$$

We note here that we use this definition of AMUC, as apposed to the equivalent definition provided by Baudier et Al. [1].

1.2.3 Graphs and Limits of Graphs

Definition 1.10 (Directed s-t Graph)**.** Presented by Baudier et Al. in [1]. A directed graph is a set of edges and vertices where each edge is connect to two vertices and has an orientation from one vertex to the next. A Directed s-t Graph is a directed graph with distinguished vertices s, t.

Definition 1.11 ($\oslash$ product of two sets)**.** Presented by Lee and Raghavendra in [8]. Let $V(H)$ be the vertex set of H, $E(H)$ be the edge set of H, $s(G)$ be the distinguished point $s \in G$, and $t(G)$ be the distinguished point $t \in G$. Given two directed s,t graphs H and G, define a new graph $H \oslash G$ as follows:

1. $V(H \oslash G) := V(H) \cup (E(H) \times (V(G) \setminus \{s(G), t(G)\}))$,
2. For every orientated edge $e = (u, v) \in E(H)$, there are $|E(G)|$ orientated edges,

$$\begin{aligned}
&\{(\{e, v_1\}, \{e, v_2\})|(v_1, v_2) \in E(G) \text{ and } v_1, v_2 \notin \{s(G), t(G)\}\} \\
&\cup \{(u, \{e, w\})|(s(G), w) \in E(G)\} \cup \{(\{e, w\}, u)|(w, s(G)) \in E(G)\} \\
&\cup \{(\{e, w\}, v)|(w, t(G)) \in E(G)\} \cup \{(v, \{e, w\})|(t(G), w) \in E(G)\}
\end{aligned}$$

3. $s(H \oslash G) = s(H)$ and $t(H \oslash G) = t(H)$.

Informally, for graphs H and G, $H \oslash G$ describes a replacement rule where every edge of H is replaced as with a copy of G. Definition 1.12 provides an example of how the $\oslash$ product is used.

The following definition gives a generalization of the example of Theorem 2.3 of [7].

Definition 1.12 (Generalized Diamond Graph)**.** Let $m = 2q$ for some $q \in \mathbb{Z}^+$ and let $k_1, k_2 \in \mathbb{Z}^+$ such that $k_1 < \frac{m}{2} < k_2$. Let (Y_i, d_{Y_i}) be the metric space Y_i with metric d_{Y_i} defined as follows:

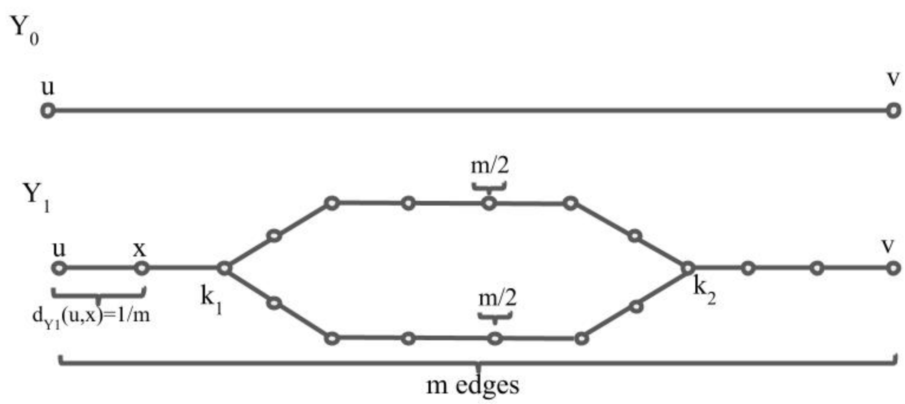
- Space:
  - For $i = 0$, define Y_0 as the graph of a single edge with two endpoints, u and v.
  - For $i = 1$ define Y_1 as in graph 1.

– For $i > 1$, define $Y_i := Y_{i-1} \oslash Y_1$.

- Metric:

  – For $y, y' \in Y_i$ for some $i \in \mathbb{Z}^+ \cup \{0\}$, let $d_Y(y, y') :=$ shortest path between y and y' in Y_i where we consider all edges to be isometric to the interval $\left[0, \frac{1}{m^i}\right]$.

Graph 1



Definition 1.13 (Hausdorff Distance and Gromov-Hausdorff Distance)**.** Presented by Heinonen in [5]. Let X and Y be non-empty subsets of a metric space (M, d). We define their Hausdorff distance, denoted $d_H(X, Y)$, as

$$d_H(X,Y) = \max\left\{\sup_{x\in X}\inf_{y\in Y} d(x,y), \sup_{y\in Y}\inf_{x\in X} d(x,y)\right\}.$$

Or equivalently

$$d_H(X,Y) = \inf\{\epsilon \geq 0 : X \subseteq Y_\epsilon \text{ and } Y \subseteq X_\epsilon\}$$

where

$$X_\epsilon = \cup_{x\in X}\{z \in M : d(z,x) \leq \epsilon\}.$$

The Gromov-Hausdorff Distance, denoted $d_{GH}(X, Y)$, is the infimum of all $d_H(f(X), g(Y))$ for all metric spaces M and isometric embeddings $f : X \to M$ and $g : Y \to M$.

Remark: The metric space $Y_\infty = \lim_{i\to\infty} Y_i$, with respect to the Gromov-Hausdorff distance, is defined as the Generalized Diamond Graph. We will show this limit exists in *Lemma 3.1.*

Definition 1.14 (Doubling Property)**.** Presented by Heinonen in [5]. Let X be a metric space with metric d_X. X is said to have the Doubling Property if there exists a doubling constant $M \in \mathbb{N}$ such that for any $x \in X$ and $r > 0$ it is possible to cover $B(x, r)$ with at most M balls with radius $\frac{r}{2}$.

Definition 1.15 (Bundle with Finite Height)**.** Presented by Baudier et Al. in [1]. A Bundle with Finite Height is a (possibly infinite) connected graph with distinguished nodes, or terminal vertices, s and t such that all simple s-t paths have equal finite length and every vertex is on a simple s-t path. The distance between s and t is called the height.

Definition 1.16 ($\mathcal{G}^\omega$)**.** Presented by Baudier et Al. in [1]. Let $\mathcal{G}^\omega$ be the family of all sequences of graphs $(G_k^\omega)_{k\in\mathbb{N}}$ satifying the following requirements:

- The base (directed) graph is G_1^ω is a bundle with finite height and infinitely many vertices.
- $G_{k+1}^\omega := G_k^\omega \oslash G_1^\omega$, for $k \geq 1$.

1.3 Main Results

1.3.1 Properties of Banach Spaces

In Section 2, we show the relationship between the quadrilateral inequality, uniform convexity, and asymptotic midpoint uniform convexity, as well as provide examples of Banach spaces that emphasize these relationships.

1.3.2 The Generalized Diamond Fractal

Theorem 3.1: Each Y_i satisfies the doubling property with doubling constant independent of i.

Corollary 3.1: There is no Bi-Lipschitz embedding from Y_∞ into a metric space Z which satisfies the quadrilateral inequality.

1.3.3 Non-Embeddability of the Countably Branching Infinite Bundle

Theorem 4.1: Let $c_Y(G_k^\omega)$ be the Y-distortion of the space G_k^ω. If Y is a Banach space admitting an equivalent AMUC norm, then $\sup_{k\in\mathbb{Z}^+} c_Y(G_k^\omega) = \infty$.

2 Properties of Banach Spaces

Lemma 2.1. Let X be a Metric space that satisfies the quadrilateral inequality. Then X also satisfies the uniform convexity property.

Proof. Recall that from Definition 1.7, X is uniformly convex if and only if $\sup_{x\in S_X} \operatorname{diam}(\operatorname{Mid}(-x,x,\delta)) \to 0$ as $\delta \to 0$. Thus let $x \in S_X$ and define arbitrary points $y_1, y_2 \in \operatorname{Mid}(-x,x,\delta)$ such that $d(y_1,y_2) = \operatorname{diam}(\operatorname{Mid}(-x,x,\delta))$. Observe by the quadrilateral inequality, we have

$$d_X(-x,y_1)^2 + d_X(x,y_1)^2 + d_X(x,y_2)^2 + d_X(-x,y_2)^2 \geq d_X(-x,x)^2 + d_X(y_1,y_2)^2.$$

Observe the following; since $-x, x \in S_X$, we have $d_X(-x,x)^2 = (2)^2 = 4$, and since $y_1, y_2 \in \operatorname{Mid}(-x,x,\delta)$, we have $d_X(x,y_1), d_X(-x,y_1), d_X(x,y_2), d_X(-x,y_2) \leq \frac{1}{2}(1+\delta)d_X(-x,x) = 1+\delta$. Therefore by equation the quadrilateral inequality, we have

$$(1+\delta)^2 \geq 1 + \frac{1}{4}d(y_1,y_2)^2 \tag{2.1}$$

or equivalently,

$$\sqrt{4((1+\delta)^2-1)} \geq \operatorname{diam}(\operatorname{Mid}(-x,x,\delta)). \tag{2.2}$$

Observe that as $\delta \to 0$, $\sqrt{4((1+\delta)^2-1)} \to 0$ and thus $\sup_{x\in S_X} \operatorname{diam}(\operatorname{Mid}(-x,x,\delta)) \to 0$. □

Lemma 2.2. Let X be a Banach space that satisfies the uniform convexity property. Then X also satisfies AMUC.

Proof. Let X be a space that satisfies the uniform convexity property. We want to show that

$$\lim_{\delta\to 0} \sup_{x\in S_X} \alpha(\operatorname{Mid}(-x,x,\delta)) = 0.$$

Observe that $\operatorname{diam}(\operatorname{Mid}(-x,x,\delta)) \geq \alpha(\operatorname{Mid}(-x,x,\delta))$. By the uniform convexity property of X we have $\sup_{x\in S_X} \operatorname{diam}(\operatorname{Mid}(-x,x,\delta)) \to 0$ as $\delta \to 0$. Therefore

$$0 \leq \lim_{\delta\to 0} \sup_{x\in S_X} \alpha(\operatorname{Mid}(-x,x,\delta)) \leq \lim_{\delta\to 0} \sup_{x\in S_X} \operatorname{diam}(\operatorname{Mid}(-x,x,\delta)) = 0$$

$\square$

Example 2.1. l_2 satisfies the quadrilateral inequality, i.e., for any points $w,x,y,z \in l_2$,

$$d_{l_2}(w,y)^2 + d_{l_2}(x,z)^2 \leq d_{l_2}(w,x)^2 + d_{l_2}(x,y)^2 + d_{l_2}(y,z)^2 + d_{l_2}(z,w)^2.$$

Proof. Let $w,x,y,z \in l_2$ and let $n \in \mathbb{N}$ be arbitrary. Observe that for the i^{th} components of $w,x,y,z \in l_2$,

$$p_i := (w_i - z_1) - (x_i - y_i) \in \mathbb{R} \text{ is such that } 0 \leq p_i^2.$$

Thus, we have

$$\begin{aligned}
0 &\leq \sum_{i=1}^{n} [(w_i - z_i) - (x_i - y_i)]^2 \\
&= \sum_{i=1}^{n} [(w_i - z_i)^2 - 2(w_i - z_i)(x_i - y_i) + (x_i - y_i)^2] \\
&= \sum_{i=1}^{n} [(w_i - z_i)^2 + (x_i - y_i)^2 - 2(-w_i y_i - x_i z_i + w_i x_i + y_i z_i)]
\end{aligned}$$

and so

$$\begin{aligned}
\sum_{i=1}^{n} (-2w_i y_i - 2x_i z_i) &\leq \sum_{i=1}^{n} [(w_i - z_i)^2 + (x_i - y_i)^2 - 2_{w_i} x_i - 2y_i z_i] \\
&= \sum_{i=1}^{n} (-2w_i x_i + x_i^2 - 2x_i y_i + y_i^2 - 2y_i z_i + z_i^2 - 2z_i w_i + w_i^2).
\end{aligned}$$

Thus by adding $\sum_{i=1}^{n} (w_i^2 + x_i^2 + y_i^2 + z_i^2)$ to both sides, we have,

$$\sum_{i=1}^{n} (w_i^2 - 2w_i y_i + y_i^2 + x_i^2 - 2x_i z_i + z_i^2)$$

$$\leq \sum_{i=1}^{n}(w_i^2 - 2w_ix_i + x_i^2 + x_i^2 - 2x_iy_i + y_i^2 + y_i^2 - 2y_iz_i + z_i^2 + z_i^2 - 2z_iw_i + w_i^2),$$

and so

$$\sum_{i=1}^{n}[(w_i - y_i)^2 + (x_i - z_i)^2] \leq \sum_{i=1}^{n}[(w_i - x_i)^2 + (x_i - y_i)^2 + (y_i - z_i)^2 + (z_i - w_i)^2],$$

and so by taking the limit as $n \to \infty$, we have

$$||w - y||_{l_2}^2 + ||x - z||_{l_2}^2 \leq ||w - x||_{l_2}^2 + ||x - y||_{l_2}^2 + ||y - z||_{l_2}^2 + ||z - w||_{l_2}^2,$$

which by definition of d, we have

$$d_{l_2}(w, y)^2 + d_{l_2}(x, z)^2 \leq d_{l_2}(w, x)^2 + d_{l_2}(x, y)^2 + d_{l_2}(y, z)^2 + d_{l_2}(z, w)^2.$$

□

Example 2.2. l_p does not satisfy the quadrilateral inequality for $p > 2$.

Proof. Suppose towards a contradiction that l_p does satisfy the quadrilateral inequality for $p > 2$. Let w, x, y, and z be the points contained in l_p such that $w = (1, 0, 0, 0, ...), x = (0, 1, 0, 0, ...), y = (-1, 0, 0, 0, ...)$ and $z = (0, -1, 0, 0, ...)$.. Thus by the quadrilateral inequality,

$$\begin{aligned} d_{l_p}(w, x)^2 + d_{l_p}(x, y)^2 + d_{l_p}(y, z)^2 + d_{l_p}(z, w)^2 &\geq d_{l_p}(w, y)^2 + d_{l_p}(x, z)^2 \\ 2^{2/p} + 2^{2/p} + 2^{2/p} + 2^{2/p} &\geq (2^p)^{2/p} + (2^p)^{2/p} \\ 4(4^{1/p}) &\geq 2(4) \\ 4^{1/p} &\geq 2 \\ 4 &\geq 2^p, \end{aligned}$$

which is a contradiction for $p > 2$. Therefore l_p does not satisfy the quadrilateral inequality for $p > 2$. □

Example 2.3. l_p is a uniformly convex Banach space for $1 < p < \infty$.

Proof. To prove Example 2.3, we reference a Corollary on page 403 of *Unifromly Convex Spaces* provided by James Clarkson [3]. □

Example 2.4. l_1 is not uniformly convex.

Proof. To show that l_1 is not uniformly convex, we will provide an example of a point $y \in l_1$ that does not satisfy the equivalent definition of uniform convexity that is $\text{diam}(\text{Mid}(-y, y, \delta)) \to 0$ as $\delta \to 0$. Thus let $y = (1/2, 1/2, 0, 0, 0, ...)$. Observe that the points $w = (-1/2, 1/2, 0, 0, 0, ...)$ and $z = (1/2, -1/2, 0, 0, 0, ...)$ are contained in $\text{Mid}(-y, y, \delta)$ and that $\|z - w\|_{l_1} = 2$. Therefore $\text{diam}(\text{Mid}(-y, y, \delta)) \geq 2$, i.e., $\text{diam}(\text{Mid}(-y, y, \delta))$ cannot go to 0 as $\delta \to 0$. □

Example 2.5. l_1 is AMUC.

Proof. Let $l_1 := \{(y_1, y_2, y_3, ...)| y_i \in \mathbb{R}, i = 1, 2, 3, ...\}$ and for $a, b \in l_1$ let $d_{l_1}(a, b) := \sum_{i=1}^{\infty} |a_i - b_i|$. We first show by a lemma that points of the form $y = (y_0, y_1, y_2, ..., y_n, 0, 0, 0, ...) \in S_{l_1}$ satisfy the AMUC property, and then use this lemma to prove that the entire space l_1 satisfies the AMUC property.

Lemma 1: If $y = (y_1, y_2, ..., y_n, 0, ...) \in S_{l_1}$, then $\text{Mid}(-y, y, \delta)$ can be covered by finitely many balls of radius 3δ.

Proof. Let $y = (y_1, y_2, y_3, ..., y_n, 0, 0, 0, ...) \in S_{l_1}$, $\delta \in (0, 1)$, and $z \in \text{Mid}(-y, y, \delta)$. Since $z \in \text{Mid}(-y, y, \delta)$, we have that

$$2\sum_{i=1}^{\infty} |z_i - y_i| \leq (1 + \delta) d_{l_1}(-y, y) = 2(1 + \delta) \tag{2.3}$$

and

$$2\sum_{i=1}^{\infty} |z_i + y_i| \leq (1 + \delta) d_{l_1}(-y, y) = 2(1 + \delta). \tag{2.4}$$

Since $y_j = 0$ for $j = n + 1, n + 2, ...$, we have by (2.3) and (2.4)

$$\sum_{i=1}^{n} |z_i - y_i| + \sum_{i=n+1}^{\infty} |z_i| \leq 1 + \delta \tag{2.5}$$

and

$$\sum_{i=1}^{n} |z_i + y_i| + \sum_{i=n+1}^{\infty} |z_i| \leq 1 + \delta. \tag{2.6}$$

Thus by adding (2.5) and (2.6) together, we have

$$\begin{aligned} 2(1+\delta) &\geq \sum_{i=1}^{n} |z_i - y_i| + \sum_{i=1}^{n} |z_i + y_i| + 2 \sum_{i=n+1}^{\infty} |z_i| \\ &\geq 2 \sum_{i=1}^{n} |y_i| + 2 \sum_{i=n+1}^{\infty} |z_i| \\ &\geq 2(1 + \sum_{i=n+1}^{\infty} |z_i|), \end{aligned}$$

hence

$$\delta \geq \sum_{i=n+1}^{\infty} |z_i|. \tag{2.7}$$

Observe that for $\sum_{i=1}^{n} |z_i|$, we have

$$\begin{aligned} \|z\|_{l_1} &\leq \|z - y\|_{l_1} + \|y\|_{l_1}, \text{ by the triangle inequality} \\ &\leq (1+\delta) + 1, \text{ since } z \in \mathrm{Mid}(-y, y, \delta) \text{ and } y \in S_{l_1} \\ &\leq 3, \end{aligned}$$

and thus

$$\sum_{i=1}^{n} |z_i| \leq 3. \tag{2.8}$$

Now by the Heine-Borel Theorem, $\bar{B}(0,3) \subset (\mathbb{R}^n, \|\cdot\|_{l_1})$ is compact. Thus, we can cover $\bar{B}(0,3)$ by finitely many balls $\bar{B}(p_k, \delta)$, where $p_k \in \mathbb{R}^n$ and $k = 1, 2, 3, ..., m$. Let $q_k := (p_k, 0, 0, ...)$, so that $q_k \in l_1$ for $k = 1, 2, 3, ..., m$. Since $z \in \mathrm{Mid}(-y, y, \delta)$, we have by (2.7) and (2.8), $\sum_{i=1}^{n} |z_i - p_{k,i}| \leq \delta$ for some k and $\sum_{i=n+1}^{\infty} |z_i - 0| \leq \delta$. Thus $\sum_{i=1}^{\infty} |z_i - q_{k,i}| \leq 2\delta$, where $q_{k,i}$ is the i^{th} coordinate of the q_k point . Hence $z \in \bar{B}(q_k, 2\delta)$, which is contained in the finite union of balls centered at q_k of radius 3δ □

Now with *Lemma 1*, suppose that $y = (y_1, y_2, y_3, ...)$ is contained in S_{l_1}, i.e., $y_j \in \mathbb{R}$ for $j = 1, 2, 3, ...$ and $\sum_{j=1}^{\infty} |y_j| = 1$. Thus for some $n \in \mathbb{Z}^+$, $\sum_{j=n+1}^{\infty} |y_j| \leq \delta$. And so set $y' =$

$(y_0, y_1, y_2, ..., y_n, 0, 0, 0, ...)$. Observe that $\|y - y'\|_{l_1} \leq \delta$. Let $y'' = \dfrac{y'}{\|y'\|_{l_1}}$ so that $y'' \in S_{l_1}$. And so

$$\begin{aligned}
\|y'' - y\|_{l_1} &\leq \|y'' - y'\|_{l_1} + \|y' - y\|_{l_1} \text{ by the triangle inequality} \\
&\leq \delta + \delta \text{ by construction of } y'' \text{ and } y' \\
&\leq 2\delta.
\end{aligned}$$

Thus,

$$\|y'' - y\|_{l_1} \leq 2\delta. \tag{2.9}$$

And so with *Lemma 1* applied to y'' and 3δ, we have that $\mathrm{Mid}(-y'', y'', 3\delta)$ is covered by finitely many balls of radius 9δ. Now observe that given $z \in \mathrm{Mid}(-y, y, \delta)$, $\|z-y\|_{l_1} \leq 1+\delta$ and $\|z+y\|_{l_1} \leq 1+\delta$, thus $\|z-y\|_{l_1} + \|y-y''\|_{l_1} \leq 1+\delta+2\delta$ and $\|z+y\|_{l_1} + \|-y+y''\|_{l_1} \leq 1+\delta+2\delta$ by (2.9), and thus by the Triangle inequality $\|z - y''\|_{l_1} \leq 1+3\delta$ and $\|z + y''\|_{l_1} \leq 1+3\delta$. Therefore $z \in \mathrm{Mid}(-y'', y'', 3\delta)$, and thus $\mathrm{Mid}(-y, y, \delta) \subset \mathrm{Mid}(-y'', y'', 3\delta)$, which is contained in finitely many balls of radius 9δ. And so diameter of these finitely many balls, ϵ, is at most 18δ, we have that

$$\lim_{\delta\to 0} \sup_{y \in S_{l_1}} \alpha(\mathrm{Mid}(-y, y, \delta)) = 0$$

for $y \in S_{l_1}$, i.e., l_1 is AMUC. □

Example 2.6. l_∞ is not AMUC.

Proof. Consider the space $l_\infty = \{(x_1, x_2, x_3, ...) : x_i \in \mathbb{R} \text{ for all } i = 1, 2, 3, ...\}$ with the norm $\|x\|_{l_\infty} := \sup_{i\in\mathbb{Z}^+} |x_i|$ and the point $y = (1, 0, 0, 0, ...)$. Let $z_1 = (0, 1, 0, 0, 0, ...)$, $z_2 = (0, 0, 1, 0, 0, ...)$, etc.. For any $i \in \mathbb{Z}^+$ $\delta > 0$, $\|z_i - y\|_{l_\infty} = 1 < 1+\delta$ and $\|z_i + y\|_{l_\infty} = 1 < 1+\delta$. Therefore, we observe that $z_i \in \mathrm{Mid}(-y, y, \delta)$ for $i = 1, 2, 3, ...$ and $\|z_j - z_k\|_{l_\infty} = 1$ for all $j \neq k$. Therefore, $\alpha(\mathrm{Mid}(-y, y, \delta)) \geq 1/2$ no matter the choice of δ, and hence l_∞ does not satisfy the AMUC property. □

3 The Generalized Diamond Graph

Lemma 3.1. The sequence of metric spaces $(Y_i)_{i\in\mathbb{Z}^+}$ converges in the Gromov-Hausdorff distance.

Proof. Observe that by construction of Y_i for any $i \in \mathbb{Z}^+$ and $j \in \mathbb{Z}^+$ such that $j > i$, $d_{GH}(Y_i, Y_{i+1}) \leq \frac{k_2 - k_1}{2m^{i+1}} \leq \frac{m-2}{m^{i+1}}$ for $2 < m \in \mathbb{Z}^+$. Thus Gromov-Hausdorff distance between Y_i and Y_j is

$$d_{GH}(Y_i, Y_j) \leq \sum_{t=0}^{j-1-i} d_{GH}(Y_{i+t}, Y_{i+t+1}) \leq \sum_{t=0}^{j-1-i} \frac{m-2}{2m^{i+t+1}}.$$

Therefore

$$d_{GH}(Y_i, Y_j) \leq \sum_{t=0}^{j-1-i} \frac{m-2}{2m^{i+t+1}} = \frac{m-2}{2m^{i+1}} \sum_{t=0}^{j-1-i} \left(\frac{1}{m}\right)^t < \frac{2(m-2)}{2m^{i+1}} = \frac{m-2}{m^{i+1}} < \frac{m}{m^{i+1}} = \frac{1}{m^i}.$$

Hence we see that $lim_{i \to \infty} \frac{1}{m^i} = 0$. Thus for a large enough i not dependent on j, the Gromov-Hausdorff distance between Y_i and Y_j can be made arbitrarily small, and thus the sequence, (Y_i), is a Cauchy sequence in the Gromov-Hausdorff distance. By Theorem 2.4 in reference [1] (Heinonen), the space of metric spaces is complete. Therefore since (Y_i) is a Cauchy sequence, it must converge to some limit, denoted as Y_∞, in the space of metric spaces. $\square$

Remark: It is important to note here that for each $i = 1, 2, 3, ...,$ Y_i contains an isometric embedding of Y_j for each $j \leq i$ for $j = 1, 2, 3, ...$

Theorem 3.1. Each Y_i satisfies the doubling property with doubling constant independent of i.

Proof. To prove that the Generalized Diamond Fractal satisfies the doubling property, we claim that there exists a number C such that, for every $i \in \mathbb{N}$, $r > 0$,and $y \in Y_i$, the ball $B(y, 2r)$ can be covered by at most C balls of radius r in Y_i. And so let $i \in \mathbb{N}$, $r > 0$,and $y \in Y_i$. Let $k = k_2 - k_1$, i.e., the number of edges between k_1 and k_2. Since the diameter of these balls less than 1, we assume $r \leq 2k$. Thus there exists a j such that $\frac{2k}{m^j} \leq r \leq \frac{2k}{m^{(j-1)}}$. Thus there exists two cases such that $i \leq j$ or $i > j$.

Case 1: Let $i \leq j$. Thus each edge of Y_i has length $\frac{1}{m^i} \geq \frac{1}{m^j} \geq \frac{r}{2km}$. And so every point in $B(y, 2r)$ is at most $4km$ edges from y. Since every vertex is connected by at most three edges, $B(y, 2r)$ has at most 3^{4km} vertices, and thus $B(y, 2r)$ is covered by at most 3^{4km} balls of radius $\frac{1}{m^i} \geq \frac{r}{2km}$, and consequently of radius r.

Case 2: Let $i > j$. Since Y_i contains an isometric embedding of Y_j, the points y of Y_j

are also contained in Y_i. By case 1, we see that $Y_j \cap B(x, 2r)$ is covered by at most $N = 3^{4km}$ balls of radius r. And so any point $z \in Y_i$ and $z \notin Y_j \cap B(x, 2r)$ is of distance at most $\frac{1}{2}k\left(\frac{1}{m^{j+1}} + \frac{1}{m^{j+2}} + ... + \frac{1}{m^i}\right) \leq \frac{k}{m^{j+1}} \leq \frac{r}{2m}$ from Y_j. Thus $B(y, 2r)$ is covered by at most N balls of radius $r + \frac{r}{2m} = \left(1 + \frac{1}{2m}\right)r = \left(\frac{1}{2} + \frac{1}{4m}\right)2r$. By the argument above, each of the balls of radius $\left(\frac{1}{2} + \frac{1}{4m}\right)2r$ is covered by at most N balls of radius at most $\left(\frac{1}{2} + \frac{1}{4m}\right)^2 2r$. So $B(x, 2r)$ is covered by at most N^2 balls of radius $\left(\frac{1}{2} + \frac{1}{4m}\right)^2 2r$. By iteration, we cover $B(y, 2r)$ by N^p balls of radius $\left(\frac{1}{2} + \frac{1}{4m}\right)^p (2r)$. Thus for p large enough depending on N, we have that $\left(\frac{1}{2} + \frac{1}{4m}\right)^p \leq \frac{1}{2}$. Hence $B(y, 2r)$ is covered by N^p balls of radius r. □

Lemma 3.2. For every $i \in \mathbb{Z}^+$, there exists an isometric embedding of Y_i into Y_∞.

Proof. Let $i \in \mathbb{Z}^+$ be fixed and consider Y_i. By construction, for each $n \geq i$ there exists an isometric embedding $\phi_n : Y_i \to Y_n$. For each n, we construct an embedding of Y_n and Y_∞ into a metric space M such that their Hausdorff distance is at most $2 * d_H(Y_n, Y_\infty) < \frac{2}{m^n}$ as determined in Lemma 1. Note that in M, we are denoting the Y_n and Y_∞ as their respective embeddings for simplicity. Thus let $\psi_n : Y_i \to Y_\infty$ such that $\psi_n(y)$ is a point such that $d_M(\psi_n(y), \phi_n(y)) < \frac{2}{m^n}$. Let $y, y' \in Y_i$. Thus

$$|d_{Y_\infty}(\psi_n(y), \psi_n(y')) - d_{Y_i}(y, y')|$$

$$\leq |d_{Y_n}(\phi_n(y), \phi_n(y')) - d_{Y_i}(y, y')| + d_M(\psi_n(y), \phi_n(y)) + d_M(\psi_n(y'), \phi_n(y')),$$

hence we have equation (1),

$$|d_{Y_n}(\phi_n(y), \phi_n(y')) - d_{Y_i}(y, y')| + d_M(\psi_n(y), \phi_n(y)) + d_M(\psi_n(y'), \phi_n(y')) \leq |0| + \frac{2}{m^n} + \frac{2}{m^n} = \frac{4}{m^n}.$$

Now let $S \subseteq Y_i$ be a dense countable subset $S = \{s_1, s_2, s_3, ...\}$. Since Y_∞ is compact, there exists a sequence $\psi_i(s_1), \psi_{i+1}(s_1), \psi_{i+2}(s_1)...$ that has a convergent sub-sequence $\{\psi_{i_j}(s_1)\}$. Again since Y_∞ is compact, for the sequence $\psi_{i_1}(s_2), \psi_{i_2}(s_2), \psi_{i_3}(s_2), ...$, there exists a sub-sequence $\{\psi_{i_{j_k}}(s_2)\}$ which converges. Proceeding in this way and diagonalizing, we obtain a sub-sequence of functions $\psi_{t_1}, \psi_{t_2}, \psi_{t_3}, ...$ such that $\psi_{t_p}(s_r)$ converges for all $s_r \in S$ as $p \to \infty$. Let $\psi(s_r) := lim_{p\to\infty}\psi_{t_p}(s_r)$. Hence by using equation (1) and for $s, s' \in S$, we have that $d_{Y_\infty}(\psi(s), \psi(s')) =$

$lim_{p\to\infty} d_{Y_\infty}(\psi_{t_p}(s), \psi_{t_p}(s')) = d_{Y_i}(s, s')$. Thus there exists an isometric embedding from S into Y_∞ which gives rise to an isometric embedding from Y_i into Y_∞. Since i was chosen arbitrarily, there exists an isometric embedding from Y_i into Y_∞ for all $i \in \mathbb{Z}^+$. $\square$

Theorem 3.2. Let $i \in \mathbb{Z}^+$ and consider Y_i defined with parameters m, k_1, k_2. Let Z be a metric space with the quadrilateral inequality and $f : Y_i \to Z$ be distance non-decreasing. Then there exists adjacent vertices a and b in Y_i for $p_i := \sqrt{1 + i\frac{4\beta^2}{m^2}}$ such that

$$p_i d_{Y_i}(a, b) \leq d_Z(f(a), f(b)),$$

where $\beta := min\left(k_2 - \frac{m}{2}, \frac{m}{2} - k_1\right) > 0$.

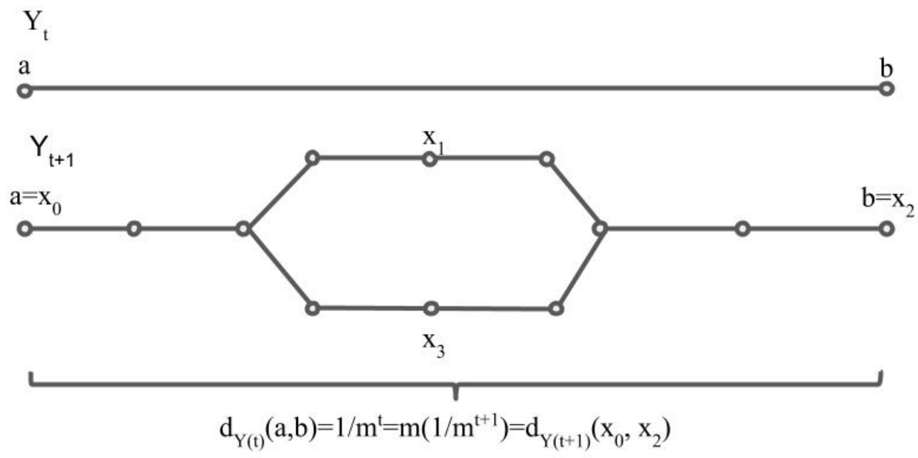
Proof. We will prove by induction that for each $i = 0, 1, 2, 3, ...$, and some pair of adjacent vertices $a, b \in Y_i$ and $p_i = \sqrt{1 + i\frac{4\beta^2}{m^2}}$, that

$$p_i d_{Y_i}(a, b) \leq d_Z(f(a), f(b)).$$

Base Case: Let $a, b \in Y_0$ be adjacent vertices. For $i = 0$, we have $p_0 = 1$, thus $p_0 d_Y(a, b) \leq d_Z(f(a), f(b))$ is $d_Y(a, b) \leq d_Z(f(a), f(b))$, which is true given that f is a distance non-decreasing function.

Inductive Step: Let $t \in \mathbb{N}$ and let $f : Y_{t+1}$ be a distance non-decreasing map into Z. Since Y_{t+1} contains an isometric copy of Y_t, $f|_{Y_t}$ is a distance non-decreasing map from Y_t into Z. By the inductive hypothesis, there exists adjacent vertices $a, b \in Y_t$ inside of Y_{t+1} such that $p_t d_{Y_t}(a, b) \leq d_Z(f(a), f(b))$. Observe that a, b are end points of a copy of Y_1 in Y_{t+1}. We define $x_0 = a, x_2 = b$ so that $x_0, x_2 \in Y_{t+1}$. Now define vertices $x_1, x_3 \in Y_{t+1}$ as two distinct midpoints of an edge that was previously in Y_t with endpoints x_0, x_2, which exist since m is defined to be even and $k_1 < \frac{m}{2} < k_2$, as in Graph 2.

Graph 2



Applying the quadrilateral inequality to the image of $f : Y_i \to Z$, for $x_4 = x_0$, we have

$$\sum_{k=0}^{3} d_Z(f(x_k), f(x_{k+1}))^2 \geq d_Z(f(x_0), f(x_2))^2 + d_Z(f(x_1), f(x_3))^2. \tag{3.1}$$

By our inductive hypothesis, we have that

$$\begin{aligned} \sum_{k=0}^{3} d_Z(f(x_k), f(x_{k+1}))^2 &\geq (p_t^2) d_{Y_{t+1}}(x_0, x_2)^2 + d_Z(f(x_1), f(x_3))^2 \\ &= \left(1 + t\frac{4\beta^2}{m^2}\right) d_{Y_{t+1}}(x_0, x_2)^2 + d_Z(f(x_1), f(x_3))^2. \end{aligned} \tag{3.2}$$

On the right hand side of (3.2), since f is a distance non-decreasing function, we have that

$$\left(1 + t\frac{4\beta^2}{m^2}\right) d_{Y_{t+1}}(x_0, x_2)^2 + d_Z(f(x_1), f(x_3))^2 \geq \left(1 + t\frac{4\beta^2}{m^2}\right) d_{Y_{t+1}}(x_0, x_2)^2 + d_{Y_{t+1}}(x_1, x_3)^2. \tag{3.3}$$

Observe that $d_{Y_{t+1}}(x_1, x_3) \geq \frac{2\beta}{m^{t+1}} = \frac{2\beta}{m} d_{Y_{t+1}}(x_0, x_2)$ given that $d_{Y_{t+1}}(x_0, x_2) = \frac{1}{m^t}$, and thus we have

$$\begin{aligned} \left(1 + t\frac{4\beta^2}{m^2}\right) d_{Y_{t+1}}(x_0, x_2)^2 + d_Y(x_1, x_3)^2 &\geq \left(1 + t\frac{4\beta^2}{m^2} + \frac{4\beta^2}{m^2}\right) d_{Y_{t+1}}(x_0, x_2)^2 \\ &\geq \left(1 + (t+1)\frac{4\beta^2}{m^2}\right) d_{Y_{t+1}}(x_0, x_2)^2 \end{aligned}. \tag{3.4}$$

Thus by (3.2),(3.3), and (3.4), we have that

$$\sum_{k=0}^{3} d_Z(f(x_k), f(x_{k+1}))^2 \geq \left(1 + (t+1)\frac{4\beta^2}{m^2}\right) d_{Y_{t+1}}(x_0, x_2)^2. \tag{3.5}$$

Now observe that by (3.5), there exists at least one $k = 0, 1, 2, 3$ such that we have

$$\begin{aligned} d_Z(f(x_k), f(x_{k+1}))^2 &\geq \frac{1}{4}\left(1 + (t+1)\frac{4\beta^2}{m^2}\right) d_{Y_{t+1}}(x_0, x_2)^2 \\ &\geq \left(1 + (t+1)\frac{4\beta^2}{m^2}\right)\left(\frac{1}{2} d_{Y_{t+1}}(x_0, x_2)\right)^2. \end{aligned} \tag{3.6}$$

Also observe that for endpoints x_0, x_2 and midpoints x_1, x_3, we have that for $k = 0, 1, 2, 3$, $d_{Y_{t+1}}(x_k, x_{k+1}) = \frac{1}{2} d_{Y_{t+1}}(x_0, x_2)$. Therefore by (3.6), we have that

$$d_Z(f(x_k), f(x_{k+1}))^2 \geq \left(1 + (t+1)\frac{4\beta^2}{m^2}\right) d_{Y_{t+1}}(x_k, x_{k+1})^2,$$

and thus

$$\begin{aligned} d_Z(f(x_k), f(x_{k+1})) &\geq \sqrt{\left(1 + (t+1)\frac{4\beta^2}{m^2}\right)}(d_{Y_{t+1}}(x_k, x_{k+1}) \\ &\geq p_{t+1} d_{Y_{t+1}}(x_k, x_{k+1}) \end{aligned}. \tag{3.7}$$

Observe that for any $k = 0, 1, 2, 3$, $d_{Y_{t+1}}(x_k, x_{k+1}) = \frac{m}{2} * \frac{1}{m^{t+1}}$. Thus there exists $\frac{m}{2} + 1$ vertices $\{x_k = v_0, v_1, ..., v_{m/2} = x_{k+1}\}$, where v_j, v_{j+1} are adjacent vertices for $j \in \mathbb{Z}^+$ and $j \in \left[0, \frac{m}{2}\right)$ on the shortest path from x_k to x_{k+1}. Thus by the construction of metric Y_{t+1},

$$d_{Y_{t+1}}(x_k, x_{k+1}) = \sum_{j=0}^{(m/2)-1} d_{Y_{t+1}}(v_j, v_{j+1}).$$

Hence

$$p_{t+1} d_{Y_{t+1}}(x_k, x_{k+1}) = p_{t+1} \sum_{j=0}^{(m/2)-1} d_{Y_{t+1}}(v_j, v_{j+1}). \tag{3.8}$$

Now observe that for the left hand side of equation (3.7), we have a path consisting of vertices

$\{x_k = v_0, v_1, ..., v_{m/2} = x_{k+1}\}$ such that by the triangle inequality,

$$\sum_{j=0}^{(m/2)-1} d_Z(f(v_j), f(v_{j+1})) \geq d_Z(f(x_k), f(x_{k+1})). \tag{3.9}$$

Hence by equations (3.7), (3.8), and (3.9), we have

$$\sum_{j=0}^{(m/2)-1} d_Z(f(v_j), f(v_{j+1})) \geq p_{t+1} \sum_{j=0}^{(m/2)-1} d_{Y_{t+1}}(v_j, v_{j+1}). \tag{3.10}$$

Thus there must exist some pair of adjacent vertices $v_j = a', v_{j+1} = b'$ where equation (3.10) implies

$$d_Z(f(a'), f(b')) \geq p_{t+1} d_{Y_{t+1}}(a', b').$$

Therefore our induction hypothesis holds for $t \in \mathbb{Z}^+$. □

Corollary 3.1. There is no Bi-Lipschitz embedding from Y_∞ into a metric space Z which satisfies the quadrilateral inequality.

Proof. Suppose towards a contradiction that there is a Bi-Lipschitz embedding $f : Y_\infty \to Z$, where Z is a metric space which satisfies the quadrilateral inequality. Without loss of generality, we may assume f is distance non-decreasing, otherwise rescale the metric on Z such that f is distance non-decreasing. Observe by *Lemma 3.2*, there exists an isometric copy of Y_i in Y_∞. By *Theorem 3.2*, for each $i \in \mathbb{N}$, there are vertices $a_i, b_i \in Y_i \subseteq Y_\infty$ such that $p_i d_{Y_\infty}(a_i, b_i) \leq d_z(f(a_i), (b_i))$, here $p_i := \sqrt{1 + i\frac{4\beta^2}{m^2}}$ and $\beta := \min\{k_2 - \frac{m}{2}, \frac{m}{2} - k_1\} > 0$. Therefore, since $p_i \to \infty$ as $i \to \infty$, there does not exist a constant C such that $d_Z(f(x)f(y)) \leq C d_{Y_\infty}(x, y)$ for all $x, y \in Y_\infty$, which is a contradiction. □

4 Non-Embeddability of the Countably Branching Infinite Bundle

The following proofs provided are based on those provided by Baudier et Al. [1], where it is shown that the countably branching infinite bundle cannot be Bi-Lipschitzly embedded into a Banach space with a norm that satisfies the AMUC property. We note that in the proofs provided for *lemma 4.2* forward that we use $\delta > 0$ as a replacement for $\tilde{\delta}(t)_Y$, where Y is a Banach space with a norm that has the asymptotically midpoint uniformly convex property and $t \in (0,1)$.

Lemma 4.1. Let X be a Banach space. Let $\delta, \lambda \in (0,1)$. Then for every $x \in X$,

$$\mathrm{Bar}_\lambda(-\lambda x, (1-\lambda)x, \delta) \subset \mathrm{Mid}(-\max\{\lambda, 1-\lambda\}x, \max\{\lambda, 1-\lambda\}x, \delta).$$

Proof. Let $z \in \mathrm{Bar}_\lambda(-\lambda x, (1-\lambda)x, \delta)$ be arbitrary. Observe that by definition of $\mathrm{Bar}_\lambda(x, y, \delta)$,

$$\mathrm{Bar}_\lambda(-\lambda x, (1-\lambda)x, \delta) = \left\{ z \in X : \max\left\{ \frac{\|z + \lambda x\|}{\lambda}, \frac{\|z - (1-\lambda)x\|}{1-\lambda} \right\} \leq (1+\delta)\|x\| \right\}.$$

Thus since $z \in \mathrm{Bar}_\lambda(-\lambda x, (1-\lambda)x, \delta)$, $\|z+\lambda x\| \leq (1+\delta)\lambda\|x\|$ and $\|z-(1-\lambda)x\| \leq (1+\delta)(1-\lambda)\|x\|$. Let $\mu := \max\{\lambda, 1-\lambda\}$. Observe that by definition of $\mathrm{Mid}(x, y, \delta)$,

$$\mathrm{Mid}(-\mu x, \mu x, \delta) = \{z \in X : \max\{2\|z + \mu x\|, 2\|z - \mu x\|\} \leq (1+\delta)2\mu\|x\|\},$$

where $(1+\delta)2\mu\|x\| = (1+\delta)\|-\mu x - \mu x\| = (1+\delta)d_X(-\mu x, \mu x)$. To show that $z \in \mathrm{Mid}(-\mu x, \mu x, \delta)$, it suffices to show that $z \in \mathrm{Mid}(-\mu x, \mu x, \delta)$ when $\mu = \lambda$ and when $\mu = 1-\lambda$.

Let $z \in \mathrm{Bar}_\lambda(-\lambda x, (1-\lambda)x, \delta)$, i.e., $\max\{\frac{\|z+\lambda x\|}{\lambda}, \frac{\|z-(1-\lambda)x\|}{1-\lambda}\} \leq (1+\delta)\|x\|)$. Suppose $\mu = \lambda$. By $z \in \mathrm{Bar}_\lambda(-\lambda x, (1-\lambda)x, \delta)$, it is immediate that

$$\|z + \lambda x\| \leq (1+\delta)\lambda\|x\|.$$

Since $z \in \mathrm{Bar}_\lambda(-\lambda x, (1-\lambda)x, \delta)$, we have that

$$\|z - (1-\lambda)x\| \leq (1+\delta)(1-\lambda)\|x\|.$$

Thus

$$\begin{aligned}
(2\lambda - 1)\|x\| &= \|(1-\lambda)x - \lambda x\| \\
&= \|(1-\lambda)x - z + z - \lambda x\| \\
&\geq \|z - \lambda x\| - \|z - (1-\lambda)x\|, \text{ by the reverse triangle inequality} \\
&\geq \|z - \lambda x\| - (1+\delta)(1-\lambda)\|x\|, \text{ since } z \in \mathrm{Bar}_\lambda(-\lambda x, (1-\lambda)x, \delta).
\end{aligned}$$

And so since $\lambda \geq \dfrac{1}{2}$ by $\mu = \lambda$, we have that

$$\begin{aligned}
\|z - \lambda x\| &\leq (2\lambda - 1)\|x\| + (1+\delta)(1-\lambda)\|x\| \\
&\leq (2\lambda - 1)\|x\| + (1+\delta)(1-\lambda)\|x\| + \delta(2\lambda - 1)\|x\|, \text{ since } \delta(2\lambda - 1)\|x\| > 0 \\
&\leq [(2\lambda - 1) + (1+\delta)(1-\lambda) + \delta(2\lambda - 1)]\|x\| \\
&= (1+\delta)\lambda\|x\|.
\end{aligned}$$

Therefore $z \in \mathrm{Mid}(-\mu x, \mu x, \delta)$ when $\mu = \lambda$.

Now suppose $\mu = 1 - \lambda$. By $z \in \mathrm{Bar}_\lambda(-\lambda x, (1-\lambda)x, \delta)$, it is immediate that

$$\|z - (1-\lambda)x\| \leq (1+\delta)(1-\lambda)\|x\|.$$

Also since $z \in \mathrm{Bar}_\lambda(-\lambda x, (1-\lambda)x, \delta)$, we have that

$$\|z + \lambda x\| \leq (1+\delta)\lambda\|x\|.$$

Thus

$$\begin{aligned}
(1 - 2\lambda)\|x\| &= \|(1-\lambda)x - \lambda x\| \\
&= \|(1-\lambda)x + z - z - \lambda x\| \\
&\geq \|z + (1-\lambda)x\| - \|z + \lambda x\|, \text{ by the reverse triangle inequality} \\
&\geq \|z + (1-\lambda)x\| - (1+\delta)\lambda\|x\|, \text{ since } z \in \mathrm{Bar}_\lambda(-\lambda x, (1-\lambda)x, \delta)
\end{aligned}$$

And so since $\lambda \leq \dfrac{1}{2}$ by $\mu = 1 - \lambda$, we have that

$$\begin{aligned}
\|z + (1-\lambda)x\| &\leq (1 - 2\lambda)\|x\| + (1+\delta)\lambda\|x\| \\
&\leq (1 - 2\lambda)\|x\| + (1+\delta)\lambda\|x\| + \delta(1 - 2\lambda)\|x\|, \text{ since } \delta(1 - 2\lambda) \geq 0 \\
&\leq [(1 - 2\lambda) + (1+\delta)\lambda + \delta(1 - 2\lambda)]\|x\| \\
&= (1+\delta)(1-\lambda)\|x\|.
\end{aligned}$$

Therefore $z \in \mathrm{Mid}(-\mu x, \mu x, \delta)$ when $\mu = 1 - \lambda$. $\square$

Lemma 4.2. If the norm of a Banach space Y is AMUC, then there is a function $\delta(t)$ such that for every $t \in (0,1)$ and every $y \in S_Y$,

$$\sup_{y \in S_Y} \alpha(\mathrm{Mid}(-y, y, \delta(t))) < t.$$

Proof. Suppose the norm of a Banach space Y is AMUC. Thus $\alpha(\mathrm{Mid}(-y,y,\delta)) \to 0$ as $\delta \to 0$. Hence for every $t \in (0,1)$, there is a $\delta(t) \in (0,1)$ such that for $\epsilon = t$, $\alpha(\mathrm{Mid}(-y,y,\delta(t))) < t$. $\square$

Lemma 4.3. If the norm of a Banach space Y is AMUC, then for every $\lambda \in (0,1)$, every $t \in (0,1)$, $\delta(t)$ from *Lemma* 4.2, and every $x, y \in Y$ where $x \neq y$, $\mathrm{Bar}_\lambda(x,y,\delta(t))$ is contained in finitely many balls of radius $t\mu\|x-y\|$, where $\mu := \max\{\lambda, 1-\lambda\}$.

Proof. Suppose that the norm of a Banach space Y is AMUC. Let $\delta := \delta(t)$ from *Lemma* 4.2 for $t \in (0,1)$, $\mu := \max\{\lambda, 1-\lambda\}$ for $\lambda \in (0,1)$, and $x, y \in Y$ such that $x \neq y$. First, we claim the following and then prove the claim at the end of the proof:

$$\mathrm{Bar}_\lambda(x,y,\delta) \subseteq (1-\lambda)x + \lambda y + \mathrm{Bar}_\lambda(-\lambda(y-x), (1-\lambda)(y-x), \delta). \tag{4.1}$$

By (4.1),

$$\mathrm{Bar}_\lambda(x,y,\delta) \subseteq (1-\lambda)x + \lambda y + \mathrm{Bar}_\lambda(-\lambda(y-x), (1-\lambda)(y-x), \delta),$$

and so with *Lemma 4.1*, we have

$$\mathrm{Bar}_\lambda(x,y,\delta) \subseteq (1-\lambda)x + \lambda y + \mathrm{Mid}(-\mu(y-x), \mu(y-x), \delta).$$

And so for points $\dfrac{y-x}{\|x-y\|_Y} \in S_Y$, we have

$$\mathrm{Bar}_\lambda(x,y,\delta) \subseteq (1-\lambda)x + \lambda y + \mu\|x-y\|_Y \mathrm{Mid}\left(-\frac{y-x}{\|x-y\|_Y}, \frac{y-x}{\|x-y\|_Y}, \delta\right)$$

Since Y is AMUC, by *Lemma 4.2* we have that for every $\dfrac{y-x}{\|x-y\|_Y} \in S_Y$,
$\mathrm{Mid}\left(-\dfrac{y-x}{\|x-y\|_Y}, \dfrac{y-x}{\|x-y\|_Y}, \delta\right)$ can be covered by finitely many sets of diameter at most t. Therefore, since $(1-\lambda)x+\lambda y$ is just a translation of the finitely many balls, we have that $\mathrm{Bar}_\lambda(x,y,\delta(t))$ is contained in finitely many balls of radius $t\mu\|y-x\|_Y$. $\square$

Proof of (4.1)**:**

Let Y be a Banach space, $x, y \in Y$ such that $x \neq y$, $\delta > 0$, and $\lambda \in (0,1)$. Suppose $z \in \mathrm{Bar}_\lambda(x,y,\delta)$, i.e., $\max\left\{\frac{\|x-z\|_Y}{\lambda}, \frac{\|z-y\|_Y}{1-\lambda}\right\} \leq (1+\delta)\|x-y\|_Y$. Let $w = z-(1-\lambda)x-\lambda y$. We wish to show that $w \in \mathrm{Bar}_\lambda(-\lambda(y-x), (1-\lambda)(y-x), \delta)$, i.e.,

$$\max\left\{\frac{\|w+\lambda(y-x)\|_Y}{\lambda}, \frac{\|(1-\lambda)(y-x)-w\|_Y}{1-\lambda}\right\} \leq (1+\delta)\|-\lambda(y-x)-(1-\lambda)(y-x)\|_Y.$$

Observe that

$$\begin{aligned}
\|-\lambda(y-x)-(1-\lambda)(y-x)\|_Y &= \|-\lambda y+\lambda x-(y-x-\lambda y+\lambda x)\|_Y \\
&= \|-\lambda y+\lambda x-y+x+\lambda y-\lambda x\|_Y \\
&= \|y-x\|_Y,
\end{aligned}$$

thus

$$\|y-x\|_Y = \|-\lambda(y-x)-(1-\lambda)(y-x)\|_Y. \tag{4.2}$$

And so since $z \in \mathrm{Bar}_\lambda(x,y,\delta)$ and by (4.2), we have

$$\begin{aligned}
(1+\delta)\|-\lambda(y-x)-(1-\lambda)(y-x)\|_Y &\geq \frac{\|x-z\|_Y}{\lambda} \\
&\geq \frac{\|x-(w+(1-\lambda)x+\lambda y)\|_Y}{\lambda} \\
&\geq \frac{\|x-w-x+\lambda x-\lambda y\|_Y}{\lambda} \\
&\geq \frac{\|-w-\lambda(y-x)\|_Y}{\lambda} \\
&= \frac{\|w+\lambda(y-x)\|_Y}{\lambda},
\end{aligned}$$

thus

$$\frac{\|w+\lambda(y-x)\|_Y}{\lambda} \leq (1+\delta)\|-\lambda(y-x)-(1-\lambda)(y-x)\|_Y. \tag{4.3}$$

Also since $z \in \mathrm{Bar}_\lambda(x, y, \delta)$ and by (4.2), we have

$$\begin{aligned}
(1+\delta)\|-\lambda(y-x)-(1-\lambda)(y-x)\|_Y &\geq \frac{\|z-y\|_Y}{(1-\lambda)} \\
&\geq \frac{\|w+(1-\lambda)x+\lambda y-y\|_Y}{(1-\lambda)} \\
&\geq \frac{\|w+x-\lambda x+\lambda y-y\|_Y}{(1-\lambda)} \\
&\geq \frac{\|w-(y-x-\lambda y+\lambda x)\|_Y}{(1-\lambda)} \\
&\geq \frac{\|w-(1-\lambda)(y-x)\|_Y}{(1-\lambda)},
\end{aligned}$$

thus

$$\frac{\|w-(1-\lambda)(y-x)\|_Y}{(1-\lambda)} \leq (1+\delta)\|-\lambda(y-x)-(1-\lambda)(y-x)\|_Y \tag{4.4}$$

Therefore by (4.3) and (4.4),

$$\max\left\{\frac{\|w+\lambda(y-x)\|_Y}{\lambda}, \frac{\|(1-\lambda)(y-x)-w\|_Y}{1-\lambda}\right\} \leq (1+\delta)\|-\lambda(y-x)-(1-\lambda)(y-x)\|_Y,$$

i.e., $w \in \mathrm{Bar}_\lambda(-\lambda(y-x), (1-\lambda)(y-x), \delta)$, which proves (4.1).

Lemma 4.4. Let G_1^ω be an infinite bundle of finite height whose terminal vertices are v_b and v_t. Let Y be a Banach space whose norm is AMUC. If $f : G_1^\omega \to Y$ is a Bi-Lipschitz embedding with distortion C, then there exists $\rho := \rho(G_1^\omega) > 0$ such that,

$$\|f(v_t) - f(v_b)\| < Lip(f)\left(1 - \frac{1}{2}\delta\left(\frac{1}{3\rho C}\right)\right) d_{G_1^\omega}(v_t, v_b).$$

Proof. Let G_1^ω be an infinite bundle with finite height whose terminal vertices are v_b and v_t. Let Y be a Banach space whose norm is AMUC. Suppose $f : G_1^\omega \to Y$ is a Bi-Lipschitz embedding with distortion C. We can also assume without loss of generality that $Lip(f) = 1$. Let $h := d_{G_1^\omega}(v_b, v_t)$ denote the height of the bundle. Since the bundle is infinite with finite height, by the pigeonhole principle there exists $k \in \mathbb{N}$ and an infinite sequence of distinct vertices $(v_j)_{j\in\mathbb{N}}$ such that $d(v_b, v_j) = k$ for all j and $d(v_j, v_t) = h - k$ for all j. Let $\rho := max\{k, h-k\}$ and $\lambda := 1 - \dfrac{k}{h}$. We claim that there exists $j \in \mathbb{N}$ such that

$$f(v_j) \notin \mathrm{Bar}_\lambda\left(f(v_t), f(v_b), \delta\left(\frac{1}{3\rho C}\right)\right). \tag{4.5}$$

Thus assuming the claim, by definition of the barycenter, we either have

$$\|f(v_j) - f(v_t)\| > \lambda\left(1 + \delta\left(\frac{1}{3\rho C}\right)\right)\|f(v_t) - f(v_b)\|, \tag{4.6}$$

or

$$\|f(v_j) - f(v_b)\| > (1-\lambda)\left(1 + \delta\left(\frac{1}{3\rho C}\right)\right)\|f(v_t) - f(v_b)\|. \tag{4.7}$$

Observe that in equations (4.6) and (4.7), $\|f(v_j) - f(v_t)\| \leq d_{G_1^\omega}(v_j, v_t) = h - k$ and $\|f(v_j) - f(v_b)\| \leq d_{G_1^\omega}(v_j, v_b) = k$ respectively. Thus if equation (4.6) holds, we have

$$\|f(v_t) - f(v_b)\| < \frac{1}{\lambda}\left(1 + \delta\left(\frac{1}{3\rho C}\right)\right)^{-1} d_{G_1^\omega}(v_j, v_t) = \frac{h}{h-k}\left(1 + \delta\left(\frac{1}{3\rho C}\right)\right)^{-1}(h-k),$$

hence

$$\|f(v_t) - f(v_b)\| < \left(1 + \delta\left(\frac{1}{3\rho C}\right)\right)^{-1} h = \left(1 + \delta\left(\frac{1}{3\rho C}\right)\right)^{-1} d_{G_1^\omega}(v_t, v_b).$$

And so if equation (4.7) holds, we have

$$\|f(v_t) - f(v_b)\| < \frac{1}{1-\lambda}\left(1 + \delta\left(\frac{1}{3\rho C}\right)\right)^{-1} d_{G_1^\omega}(v_j, v_b) = \frac{h}{k}\left(1 + \delta\left(\frac{1}{3\rho C}\right)\right)^{-1} k,$$

hence

$$\|f(v_t) - f(v_b)\| < \left(1 + \delta\left(\frac{1}{3\rho C}\right)\right)^{-1} h = \left(1 + \delta\left(\frac{1}{3\rho C}\right)\right)^{-1} d_{G_1^\omega}(v_t, v_b).$$

Therefore, both equations (4.5) and (4.7) imply

$$\|f(v_t) - f(v_b)\| < \left(1 + \delta\left(\frac{1}{3\rho C}\right)\right)^{-1} d_{G_1^\omega}(v_t, v_b). \tag{4.8}$$

Now observe that for $\delta\left(\frac{1}{3\rho C}\right) \in (0,1)$, we have that $\frac{1}{2}\delta\left(\frac{1}{3\rho C}\right) > \frac{1}{2}\delta\left(\frac{1}{3\rho C}\right)^2$, thus

$$\begin{aligned} 1 &< 1 + \frac{1}{2}\delta\left(\frac{1}{3\rho C}\right) - \frac{1}{2}\delta\left(\frac{1}{3\rho C}\right)^2 \\ &= \left(1 - \frac{1}{2}\delta\left(\frac{1}{3\rho C}\right)\right)\left(1 + \delta\left(\frac{1}{3\rho C}\right)\right). \end{aligned}$$

Therefore $\left(1 + \delta\left(\frac{1}{3\rho C}\right)\right)^{-1} < \left(1 - \frac{1}{2}\delta\left(\frac{1}{3\rho C}\right)\right)$, and thus by equation (4.8),

$$\begin{aligned} \|f(v_t) - f(v_b)\| &\leq 1 \cdot \left(1 + \delta\left(\frac{1}{3\rho C}\right)\right)^{-1} d_{G_1^\omega}(v_t, v_b) \\ &< Lip(f)\left(1 - \frac{1}{2}\delta\left(\frac{1}{3\rho C}\right)\right) d_{G_1^\omega}(v_t, v_b). \end{aligned}$$

Proof of (4.5): By *Lemma 4.3*, $\mathrm{Bar}_\lambda\left(f(v_t), f(v_b), \delta\left(\frac{1}{3\rho C}\right)\right)$ is contained in finitely many balls of radius $\frac{1}{3\rho C} max\{\lambda, 1-\lambda\}\|f(v_t) - f(v_b)\| \leq \frac{1}{3\rho C} max\{\lambda, 1-\lambda\} h = \frac{1}{3C}$. Observe by $f : G_1^\omega \to Y$ with distortion C and Bi-Lipschitz constant $Lip(f) = 1$, we have that $d_Y(f(v_i), f(v_j)) \geq \frac{1}{C} d_{G_1^\omega}(v_i, v_j)$ for any $v_i, v_j \in G_1^\omega$. Also since $d_{G_1^\omega}(v_i, v_j) \geq 1$, we have $d_Y(f(v_i), f(v_j)) \geq \frac{1}{C}$. Therefore, given balls of radius $\frac{1}{3C} < \frac{1}{2C}$, we have that each ball contains at most one point $f(v_i) \in Y$. Since there are infinitely many $f(v_i) \in Y$, there must be infinitely balls of radius $\frac{1}{3C}$ to contain $\mathrm{Bar}_\lambda\left(f(v_t), f(v_b), \delta\left(\frac{1}{3\rho C}\right)\right)$, unless some $f(v_j) \notin \mathrm{Bar}_\lambda\left(f(v_t), f(v_b), \delta\left(\frac{1}{3\rho C}\right)\right)$. □

Proposition 4.1. Let $(G_k^\omega)_{k\in\mathbb{Z}^+} \in G^\omega$, and Y be an asymptotically midpoint uniformly convex Banach space. There exists $\rho > 0$, such that if $k \in \mathbb{N}$ and G_k^ω embeds Bi-Lipschitzly into Y with distortion C, then G_{k-1}^ω embeds Bi-Lipschitzly into Y with distortion at most $C\left(1 - \frac{1}{2}\delta\left(\frac{1}{3\rho C}\right)\right)$.

Proof. Let Y be a Banach space with an AMUC norm. Let $f_k : G_k^\omega \to Y$ be a C-Lipschitz function, i.e., $d_{G_k^\omega}(a,b) \leq d_Y(f(a), f(b)) \leq C d_{G_k^\omega}(a,b)$ for $a, b \in G_k^\omega$. Without loss of generality, we assume f_k is a distance-non decreasing function. Let $V(G_k^\omega)$ denote the set of vertices in G_k^ω. Thus observe that $V(G_{k-1}^\omega) \subset V(G_k^\omega)$, since $G_k^\omega := G_{k-1}^\omega \oslash G_1^\omega$, which forms a copy up to a scale of $h(G_k^\omega)$. Let $g_k := f_k|_{V(G_{k-1}^\omega)}$ re-scaled by $h(G_k^\omega)$ and let $v_{t_{k-1}}, v_{b_{k-1}} \in G_{k-1}^\omega$ be adjacent vertices. Observe that

$v_{t_{k-1}} = v_t$ and $v_{b_{k-1}} = v_b$ for a copy of G_1^ω scaled by a factor of $h(G_1^\omega)$. Thus by Lemma 4.4,

$$\|g_k(v_{t_{k-1}}) - g_k(v_{b_{k-1}})\| \le C\left(1 - \frac{1}{2}\delta\left(\frac{1}{3\rho C}\right)\right) d_{G_k^\omega}(v_{t_{k-1}}, v_{b_{k-1}}).$$

For general $v_i, v_j \in G_{k-1}^\omega \subset G_k^\omega$, there exists a path $v_i = v_0, v_1, v_2, ..., v_n = v_j$ consisting of adjacent vertices in G_{k-1}^ω such that

$$d_{G_k^\omega}(v_i, v_j) = \sum_{r=0}^{j-1} d_{G_k^\omega}(v_r, v_{r+1}).$$

Therefore

$$\begin{aligned} d_{G_k}(v_i, v_j) \le \|g_k(v_i) - g_k(v_j)\| &\le \sum_{r=0}^{j-1} \|g_k(v_r) - g_k(v_{r+1})\|, \text{ by the triangle inequality} \\ &\le C\left(1 - \frac{1}{2}\delta\left(\frac{1}{3\rho C}\right)\right) \sum_{r=0}^{j-1} d_{G_k^\omega}(v_r, v_{r+1}), \text{ by } \textit{Lemma 4.4} \\ &\le C\left(1 - \frac{1}{2}\delta\left(\frac{1}{3\rho C}\right)\right) d_{G_k^\omega}(v_i, v_j). \end{aligned}$$

Therefore G_{k-1}^ω embeds Bi-Lipschitzly into Y with distortion at most $C\left(1 - \frac{1}{2}\delta\left(\frac{1}{3\rho C}\right)\right)$. □

Theorem 4.1. Let $c_Y(G_k^\omega)$ be the Y-distortion of the space G_k^ω. If Y is a Banach space admitting an equivalent AMUC norm, then $\sup_{k\in\mathbb{Z}^+} c_Y(G_k^\omega) = \infty$.

Proof. Suppose Y is a Banach space admitting an equivalent asymptotically midpoint uniformly convex norm. Let $c_Y(G_k^\omega)$ be the Y-distortion of the space G_k^ω. Suppose to the contrary that $c_Y(G_k^\omega) \le C < \infty$ for all $k \in \mathbb{Z}^+$ and some constant C. Let $\eta := 1 - \frac{1}{2}\delta\left(\frac{1}{3\rho C}\right)$ and observe that $\eta \in (0, 1)$. Let $r \in (1, \frac{1}{\eta})$. For each k, there exists an embedding $f_k : G_k^\omega \to Y$ with distortion $C_k \le r c_Y(G_k^\omega)$. Thus using *Proposition 4.1*, we have that

$$\frac{1}{r} C_{k-1} \le c_Y(G_{k-1}^\omega) \le C_k \eta.$$

So $\frac{1}{r\eta} C_{k-1} \le C_k$ for all k. Since $\frac{1}{r\eta} > 1$, $C_k \to \infty$ as $k \to \infty$. Therefore since $c_Y(G_k^\omega) \ge \frac{1}{r} C_k$ for all k, we have $c_Y(G_k^\omega) \to \infty$ as $k \to \infty$. This is a contradiction. □

5 References

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